PLC implementations of an elementary Fractional Order operator

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Abstract

The proposed paper is intended to show a PLC implementation of an elementary fractional order, integro-differential operator. The considered element is approximated with the use of known discrete PSE and CFE approximations. It is a main part of fractional order FO models and control algorithms, for example FO PID controller. To implement SIEMENS SIMATIC S7 1200 and 1500 platforms were employed. The both proposed approximations PSE and CFE were compared in the sense of accuracy, convergence and execution time. Results of experiments show, that the PLC implementation of the fractional order element can be done with the use of object-oriented approach, the accuracy of each approximation is determined by its order. The CFE approximation is much more faster that PSE, but its accuracy is a little bit lower.

1 An Introduction

Main areas of application the fractional order calculus in automation are: fractional order control and modeling of processes with dynamics hard to describe with the use of another approaches. Fractional order control covers mainly particularly Fractional Order PID controllers (FO PID). FO PID controllers have been presented by many Authors and their usefulness has been proven (see for example: [4], [7], [22], [24], [20]). A PLC implementation of FO controller was presented for example in [23].

However, the practical implementation of FO controllers and models causes a number of problems, generated mainly by the fact, that the fractional order differentiation/integration operator is impossible to exact implementation and it requires to use approximations, possible to digital implementation. It can be done with the use of PSE (Power Series Expansion), CFE (Continuous Fraction Expansion) approximation or discrete version of ORA (Ostaloup Recursive Approximation) approximation .

This paper is intented to show possibilities of implementation a basic Fractinal Order (FO) element s^α at PLC. The considered element is the basic brick to implement many fractional order controllers and models at PLC. To implement the most typical PSE and CFE approximations were employed, results were collected using SCADA. Experiments cover tests of accuracy and execution time during evaluation of FO calculations.

The paper is organized as follows: at the beginning any elementary ideas from non integer order calculus are given, particularly the both applied discrete approximations are presented. Next the experimental results are given and finally main conclusions are formulated.

2 Preliminaries

2.1 Elementary ideas

The presentation of elementary ideas will be started with define a non integer order, integro-differential operator. It is expressed as follows (see for example [12]):

$$ {}_0D_t^\alpha f(t) = \begin{cases} \frac{d^\alpha f(t)}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \int\limits_a^t f(\tau)(d\tau)^{-\alpha} & \alpha < 0 \end{cases} . \tag{1} $$

where a and t denote time limits to operator calculating, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation.
Next an idea of Gamma Euler function (see for example [13]) can be given:

$$ \Gamma(x) = \int\limits_0^\infty t^{x-1} e^{-t} dt. \tag{2} $$

The fractional-order, integro-differential operator (1) can be described by different definitions, given by Grünvald and Letnikov (GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). The digital modeling of FO operator can be most naturally done with the use of GL definition and it will be presented here:

$$ {}_{0}^{GL}D_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t}{h}]} (-1)^j \binom{\alpha}{j} f(t - jh). \tag{3} $$

In (3) $\binom{\alpha}{j}$ is a generalization of Newton symbol into real numbers:

$$ \binom{\alpha}{j} = \begin{cases} 1, & j = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}, & j > 0 \end{cases} \tag{4} $$

The transfer funcion of the elementary operator (1) described by Caputo definition has very simple and intuitive form. It is equal directly s^{α}. The analytical form of the step response for this element is expressed as underneath:

$$ y_{an}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{5} $$

The analytical formula of step response (5) will be applied as a standard to estimate an accuracy of the both tested PLC implementations.
An implementation of operator (1) at each digital platform (PLC, microcontroller) requires us to apply its integer order, finite dimensional, discrete approximant. The most known are PSE (Power Series Expansion) and CFE (Continuous Fraction Expansion). They allow us to estimate a non integer order element with the use of digital FIR or IIR filter.

2.2 The PSE approximation

The PSE (Power Series Expanasion) approximation derives directly from GL definition. Its discrete verison is called Fractional Order Backward Difference (FOBD) for $\alpha > 0$ and analogically Fractional Order Backward SUM (FOBS) for $\alpha < 0$:

$$ (\Delta^{\alpha} x)(t) = \frac{1}{h^{\alpha}} \sum_{l=0}^{\infty} (-1)^l \binom{\alpha}{l} x(t - lh). \tag{6} $$

Let us denote coefficients $(-1)^l \binom{\alpha}{l}$ by d_l:

$$ d_l = (-1)^l \binom{\alpha}{l}. \tag{7} $$

The coefficients (7) can be also calculated with the use of the following, equivalent recursive formula (see for example [4], p. 12). This formula will be applied to calculate coefficients at PLC platform during experiments presented in the next section.

$$d_0 = 1$$
$$d_l = \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \quad l = 1, ..., L. \tag{8}$$

In reality the calculation of sum (6) is possible for finite values of l only. The maximal value of l is known as memory length L and the finite approximation of (6) turns to the following form:

$$(\Delta^\alpha x)(t) = \frac{1}{h^\alpha} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} x(t - lh). \tag{9}$$

In (9) L denotes a memory length necessary to correct approximate of a non integer order operator. Unfortunately the good accuacy of PSE approximation requires us to use long memory L what can make difficulties during digital implementation at bounded platform. The FOBD (9) can be also expressed as discrete FIR filter containing zeros only:

$$(\Delta^\alpha x)(t) = \frac{1}{h^\alpha} \sum_{l=0}^{L} d_l z^{-l}. \tag{10}$$

The time response of the above approximant can be easily calculated as follows:

$$y^+_{PSE}(k) = \frac{1}{h^\alpha} \sum_{l=0}^{L} d_l u^+(k - l). \tag{11}$$

where $y^+_{PSE}(k)$ is the output in k time step, $u^+(k-l)$ denote the input signals in $k - l$ - th time moment, d_l are coefficients of PSE approximation, given by (8). The equation (11) will be directly implemented as function block (FB) at PLC. The use of FB is caused by the fact, that the correct calculation of (11) requires us to know L previous steps of output and a FB is the smallest Program Organization Unit (POU) assuring the "memory function" for its variables. Unfortunately, the value of L assuring the sensible accuracy of this approximant needs to be long (typically greater that 100). This fact can cause problems during real time implementation and it needs to be tested to avoid time errors. Results of such tests will be given in the next section.

2.3 The CFE approximation

An alternative approach during modeling a FO operator is to use CFE approximation. The CFE model has the shape of IIR filter containing both poles and zeros. It is faster covergent and easier to implement because its useful order is relatively low, typically not higher that 5.

The discretization of fractional order element s^α, $\alpha \in \mathbb{R}$ can be done with the use of the so called generating function $s \approx \omega(z^{-1})$. The new operator raised to power α has the following form (see for example [5], [22], p.119):

$$\begin{aligned}(\omega(z^{-1}))^{\alpha} &= \left(\tfrac{1+a}{h}\right)^{\alpha} CFE\{\left(\tfrac{1-z^{-1}}{1+az^{-1}}\right)^{\alpha}\}_{M,M} = \\ &= \tfrac{P_{\alpha M}(z^{-1})}{Q_{\alpha M}(z^{-1})} = \left(\tfrac{1+a}{h}\right)^{\alpha} \tfrac{CFE_N(z^{-1},\alpha)}{CFE_D(z^{-1},\alpha)} = \frac{\sum\limits_{m=0}^{M} w_m z^{-m}}{\sum\limits_{m=0}^{M} v_m z^{-m}}\end{aligned} . \quad (12)$$

In (12) a is the coefficient depending on approximation type (for example: a=0 for Euler approximation, a=1 for Tustin approximation), h denotes the sample time, M is the order of approximation. Numerical values of coefficients w_m and v_m and different values of parameter a can be calculated for example with the use of MATLAB function given by Petras in [25]. This MATLAB function was applied in experiments described in the next section. If the Tustin approximation is considered (a=1) then $CFE_D(z^{-1}, \alpha) = CFE_N(z^{-1}, -\alpha)$ and the polynomial $CFE_D(z^{-1}, \alpha)$ can be given in the direct form (see [5]). Examples of polynomial $CFE_D(z^{-1}, \alpha)$ for $M = 1, 3, 5$ are given in table 1.

Table 1: coefficients of CFE polynomials $CFE_{N,D}(z^{-1}, \alpha)$ for Tustin approximation with respect to [5].

Order M	w_m	v_m
M=1	$w_1 = -\alpha$ $w_0 = 1$	$v_1 = \alpha$ $v_0 = 1$
M=3	$w_3 = -\frac{\alpha}{3}$ $w_2 = \frac{\alpha^2}{3}$ $w_1 = -\alpha$ $w_0 = 1$	$v_3 = \frac{\alpha}{3}$ $v_2 = \frac{\alpha^2}{3}$ $v_1 = \alpha$ $v_0 = 1$
M=5	$w_5 = -\frac{\alpha}{5}$ $w_4 = \frac{\alpha^2}{5}$ $w_3 = -\left(\frac{\alpha}{5} + \frac{2\alpha^3}{35}\right)$ $w_2 = \frac{2\alpha^2}{5}$ $w_1 = -\alpha$ $w_0 = 1$	$v_5 = \frac{\alpha}{5}$ $v_4 = \frac{\alpha^2}{5}$ $v_3 = -\left(\frac{-\alpha}{5} + \frac{-2\alpha^3}{35}\right)$ $v_2 = \frac{2\alpha^2}{5}$ $v_1 = \alpha$ $v_0 = 1$

The time response of the approximated FO element (12) in k-th time moment is expressed as underneath:

$$y^{+}_{CFE}(k) = \frac{1}{v_0}\left[-\sum_{m=1}^{M} v_m y^{+}(k-m) + \sum_{m=0}^{M} w_m u^{+}(k-m)\right]. \quad (13)$$

where $y^{+}_{CFE}(k-m)$ and $u^{+}(k-m)$ denote the output and input signals in $k-m$ - th time moments respectively, v_m and w_m are coefficients of CFE approximation, given in the table 1. The equation (13) will be directly implemented as function block (FB) at PLC.

2.4 The cost function

The accuracy of the both considered approximations (9) or (13) will be estimated with the use of typical MSE (Medium Square Error) cost function:

$$MSE = \frac{1}{K_s} \sum_{k=1}^{K_s} \left(y(kh) - y^{+}_{CFE/PSE}(k) \right)^2 . \tag{14}$$

where K_s is a number of all collected samples, y is the analytical time response calculated in discrete time steps kh, $y^{+}_{CFE/PSE}$ is the time response of CFE/PSE approximation, calculated at PLC along the same time grid and with respect to (11) or (13). If we assume that the input signal $u(t)$ is a Heviside function: $u(t) = 1(t)$, then $y(t) = y_{an}(t)$, where y_{an} is expressed by (5).

3 Experiments

3.1 The PSE implementation

PSE model was implemented at budget SIEMENS S7 1200 PLC system. It contains the following elements: PLC SIEMENS 1200 with CPU 1212C, HMI panel SIEMENS KTP400 and industrial switch CSM1277. The system is connected to PC with software SIEMENS TIA PORTAL V13 via PROFINET. All parameters to experiments were introduced via HMI, it was employed also to store results at pendrive in text format.

The software implementing the tested FO element was prepared with the use of standard elements available at TIA PORTAL v13 platform. All elements of program were connected via PLC tags, which are equivalent to Directly Represented Variables described by IEC61131.3 standard.

The performance of PSE approximation was estimated with the use of MSE cost function (14). Tests were done for different values of fractional order α and different memory lengths: $L = 10, 20, 30, 40, 50$. The PSE apprixmation was evaluated with the sample time h equal 1[s], the number of collected samples K_s was equal 50. The positive values of α (differentiator) and negative values (integrator) were tested separately. Values of cost function (14) for all tests are given in tables 2 and 3, diagrams of step responses are given in table 4.

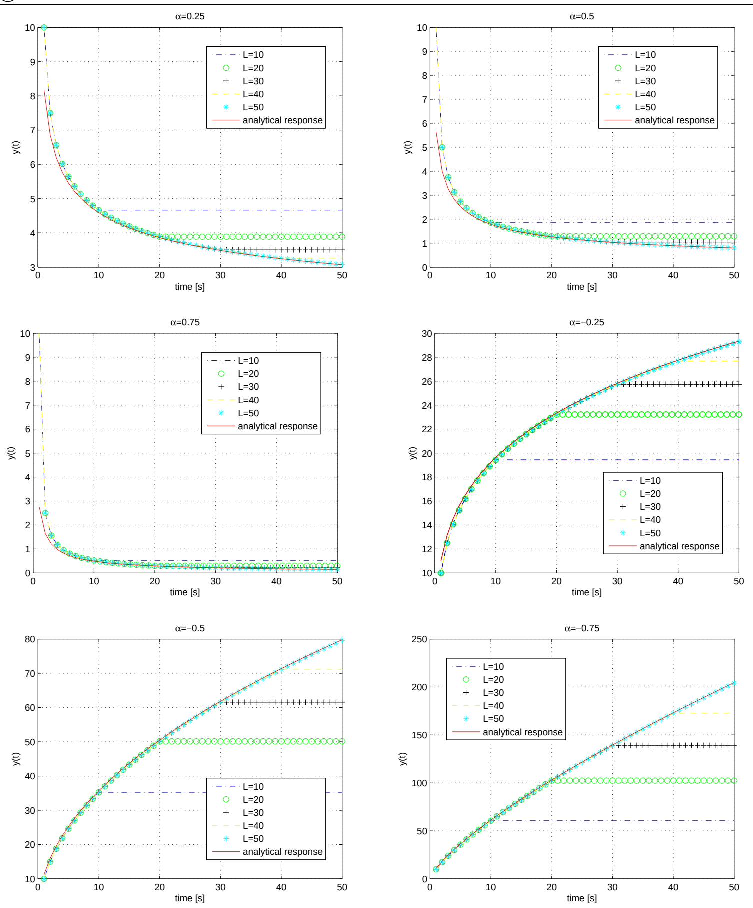
Table 2: MSE cost function (14) for PSE approximation and positive values of α

	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
L=10	1.1664	0.9246	1.1387
L=20	0.2680	0.4788	1.0750
L=30	0.1145	0.4201	1.0686
L=40	0.0850	0.4104	1.0678
L=50	0.0820	0.4095	1.0677

Table 3: MSE cost function (14) for PSE approximation and negative values of α

	$\alpha = -0.25$	$\alpha = -0.50$	$\alpha = -0.75$
L=10	35.4139	655.6779	6265.6482
L=20	9.3249	206.3641	2318.2036
L=30	2.1109	51.3412	638.5235
L=40	0.2757	6.1833	80.5312
L=50	0.0501	0.1685	0.2527

Table 4: The step responses: analytical (5) and PSE approximated for different orders α and memory length L



3.2 The CFE implementation

The hardware and software for CFE approximant was prepared analogically as for the PSE presented in the previous section. The performance of CFE approximation was estimated with the use of MSE cost function(14) also. Tests were executed for different values of fractional order α and different values of CFE approximation order $M = 1 \ldots 5$. The Tustin apprixmation was applied ($a = 1$) with the sample time h equal 1[s], the number of collected samples K_s was equal 50. The positive values of α (differentiator) and negative values (integrator) were tested separately also. Values of cost function (14) for all tests are given in tables 5 and 6, diagrams of step responses are given in table 7.

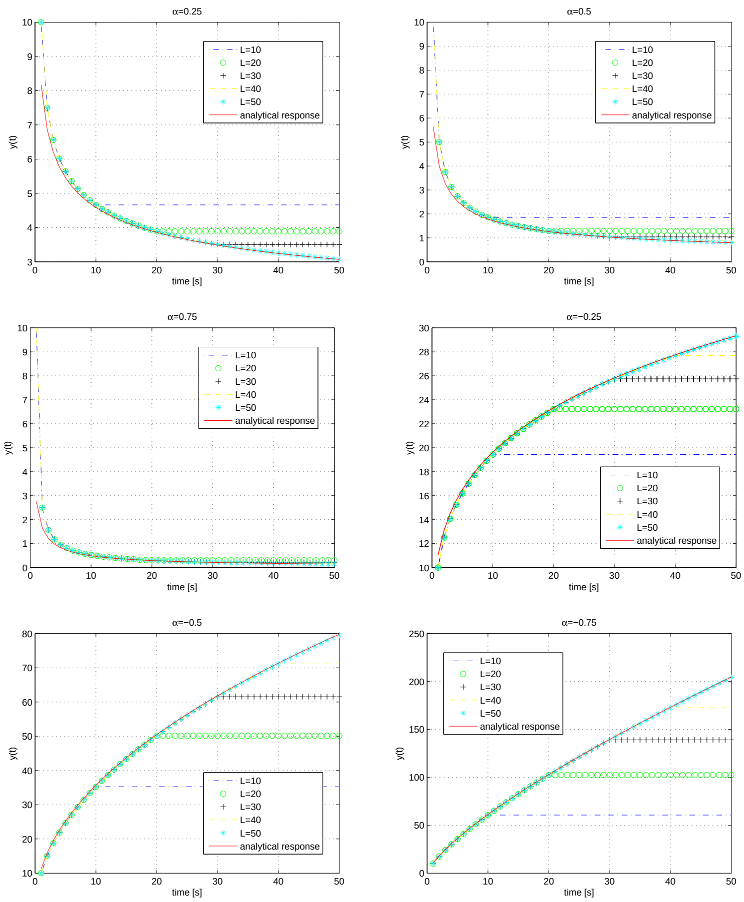
Table 5: MSE cost function (14) for positive values of α

	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
M=1	5.0838	4.6758	2.2865
M=2	1.1845	1.1265	1.2097
M=3	0.3229	0.5522	1.0920
M=4	0.1256	0.4357	1.0720
M=5	0.0885	0.4136	1.0690

Table 6: MSE cost function (14) for negative values of α

	$\alpha = -0.25$	$\alpha = -0.50$	$\alpha = -0.75$
M=1	69.0932	899.0891	5281.0921
M=2	19.7334	235.8201	906.2536
M=3	4.1168	40.2426	109.7010
M=4	0.6651	5.3124	11.6897
M=5	0.1263	0.7404	1.4017

Table 7: The step responses: of the plant (red line) and of model with different α and different orders M of CFE approximant



3.3 The real time tests

Real-time experiments were done using the simple S7 1500 PLC system containing CPU 1511-1 PN (no 6ES7 511-1AK00-0AB0) with firmware 1.8, connected to PC with software SIEMENS TIA PORTAL V13 via PROFINET. All parameters to experiments were introduced via simple SCADA application implemented at PC using WinCC software. It was employed also to collect results. The input and output modules were not applied because they were not neccessary to run all the tests. Tests were done for different values of fractional order α and different approximation orders: $L = 20..200$ for PSE and $M = 1...5$ for CFE. The sample time h was equal 1[s], the number of collected samples K_s was equal 50. The positive values of α (differentiator) and negative values (integrator) were tested separately. Values of measured durations are given in tables: 8, 9, 10 and 11.

Table 8: Cycle times during execution the **PSEcoeff** function (calculation of coefficients d_l for PSE approximation) in [ms]

α/L	10	20	30	40	50	100	200
0.25	0,9	0,9	0,9	0,9	1	0,9	1
0.5	0,9	0,9	0,9	0,9	0,9	1	1
0.75	0,9	1	1	0,9	1	1	1
-0.25	0,9	0,9	0,9	1	0,9	1	1
-0.5	0,9	0,9	1	1	1	1	1
-0.75	0,9	0,9	1	1	1	1	1

Table 9: Cycle times during execution of instance **PSE** function block (PSE approximant calculation) at OB30 in [ms]

α/L	10	20	30	40	50	100	200
0.25	3,7	6,2	8,0	9,1	9,5	36,3	143,2
0.5	3,8	6,3	8,1	9,1	9,6	36,5	143,4
0.75	3,8	6,3	8,2	9,2	9,5	36,5	143,4
-0.25	3,7	6,3	8,1	9,1	9,5	36,4	143,3
-0.5	3,8	6,4	8,2	9,2	9,6	36,5	143,4
-0.75	3,8	6,3	8,1	9,3	9,5	36,6	143,5

Table 10: Cycle times during during execution the **CFEcoeff** function (calculation of coefficients w_m andv_m for CFE approximation in [ms]

α/M	1	2	3	4	5
0.25	0,61	0,63	0,67	0,69	0,77
0.5	0,61	0,63	0,67	0,69	0,76
0.75	0,62	0,63	0,68	0,69	0,76
-0.25	0,63	0,64	0,68	0,69	0,77
-0.5	0,63	0,65	0,68	0,7	0,78
-0.75	0,62	0,63	0,67	0,68	0,77

Table 11: Cycle times during execution of **CFE** function block instance (calculating of CFE approximation) at OB30 in [ms]

α/M	1	2	3	4	5
.25	0,15	0,15	0,15	0,14	0,14
0.5	0,15	0,14	0,14	0,14	0,14
0.75	0,15	0,14	0,14	0,14	0,14
-0.25	0,14	0,14	0,14	0,14	0,14
-0.5	0,15	0,14	0,14	0,14	0,14
-0.75	0,15	0,14	0,14	0,14	0,14

4 Final Conclusions

Final conclusions from the paper can be formulated as underneath:

- The elementary fractional order plant, expressed by s^α transfer function can be implemented at PLC platform with the use of normalized software tools,

- the accuracy of model is determined by the order of approximation: higher order gives the better accuracy,

- the use of CFE approximant allows us to obtain properly working fractional order element with sensible order and short execution time. This can be pointed as advantage of this method in contrast to PSE approximation, where similar accuracy requires us to use much more higher (and complex) model,

- the cycle time during execution FO calculations strongly depends on order of approximation. This implies that the CFE approximation is much more faster that PSE with huge memory length necessary to obtain the good accuracy.

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